p-FACTORABLE OPERATORS

BY

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ABSTRACT. Several classes of operators on Banach spaces, defined by certain summability conditions on the kth approximation numbers, are introduced and studied. Characterizations of these operators in terms of tensor-product representations are obtained. The relationship between these operators and other classes of operators introduced by various authors is studied in some detail.

Introduction. This paper is a study of a class of operators on Banach spaces which we call the p-factorable operators. The notion of a p-factorable operator is a natural extension to the operators of type l^p of ideals of Grothendieck concerning the factorization of nuclear operators. Since it is easier to study properties of diagonal operators on the l^p spaces, it is useful to know which operators admit such factorizations. We give a characterization of the p-factorable operators in terms of certain tensor product representations. Such representations are useful, for example, when studying the summability properties of the eigenvalues of an operator. The classical theorem of H. Weyl discussed in §1 is an example of this. In §2 we study in some detail the relationship between the p-factorable operators and several other classes of operators on Banach spaces which have been introduced by several authors.

We begin by giving basic definitions and establish the notation which will be used throughout this paper.

All spaces considered are Banach spaces. We denote the unit ball of E by U_E and the space of all continuous linear functionals on E by E'. By operator we mean a bounded linear transformation. The collection of all operators from E to F will be denoted by $L(E, F)$ and $\|T\|$ denotes the usual operator norm for $T \in L(E, F)$. For $T \in L(E, F)$ the astriction of T, T_a, is the operator $T_a: E \longrightarrow \overline{T(E)}$ defined by $T_a x = Tx$ for all $x \in E$. If $f \in E'$ and $y \in F$ then by $f \otimes y$ we mean the rank one operator $f \otimes y: E \longrightarrow F$ defined by $f \otimes y(x) = \langle x, f \rangle y$ for every $x \in E$.

We will be interested in the following classes of operators:

The operators of type l^p. For $T \in L(E, F)$ the *kth approximation number of T, $\alpha_k(T)$,* is defined by $\alpha_k(T) = \inf\|T - A\|$, the inf being taken over all

$A \in L(E, F)$ of rank at most k. For detailed information concerning properties of the kth approximation numbers and their relationship to other approximation numbers associated with operators on Banach spaces see [2], [5], [6], [7] and [8].

Following Pietsch we say an operator $T \in L(E, F)$ is of type l^p, $0 < p \le \infty$, if the sequence $(\alpha_k(T))_{k=0}^{\infty}$ belongs to l^p (or to c_0 if $p = \infty$). We denote the collection of operators of type l^p from E to F by $l^p(E, F)$.

The strongly p-summable operators. In the second chapter of his memoir, Grothendieck [1] introduced a class of operators on Banach spaces which he called operators "de puissance p. ème sommable" for $0 < p \le 1$. To avoid confusion with other classes of operators we shall refer to these operators as strongly p-summable. Namely $T \in L(E, F)$ is said to be *strongly p-summable*, $0 < p \le 1$, if T has a representation $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ where $(\lambda_i)_{i=1}^{\infty} \in l^p$, $(f_i)_{i=1}^{\infty} \subset U_{E'}$, and $(y_i)_{i=1}^{\infty} \subset U_F$. Following Grothendieck we will let $L^{(p)}(E, F)$ denote the complete, metrizable, topological vector space of all strongly p-summable operators from E to F given the topology generated by

$$S_p(T) = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i|^p : T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i \right\}.$$

The operators of Markus. The class of operators $F_p(E, F)$, $0 < p \le 1$, was introduced and studied by Markus in [4] and is defined to be the collection of all $T \in L(E, F)$ having a representation $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ with $(f_i)_{i=1}^{\infty} \subset U_{E'}$, $(y_i)_{i=1}^{\infty} \subset U_F$ and $(\lambda_i)_{i=1}^{\infty} = o(i^{-1/p})$. Given the topology generated by

$$F_p(T) = \inf \left\{ \sup_n n^{1/p} |\lambda_n| : T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n \right\},$$

$F_p(E, F)$ is complete and metrizable. For results pertaining to the relation between $F_p(E, F)$ and various other classes of operators see [4].

1. *p-factorable operators.* In his classic memoir [1], Grothendieck showed that a nuclear operator on arbitrary Banach spaces factors through a nuclear diagonal operator from l^{∞} to l^1 (by a diagonal operator $T \sim (\lambda_i)_{i=1}^{\infty}$ between sequence spaces we mean the operator $T(\xi_i)_{i=1}^{\infty} = (\lambda_i \xi_i)_{i=1}^{\infty}$ where the λ_i are scalars). Since every operator of type l^1 is nuclear [6], the question naturally arises whether operators of type l^1 factor in a similar manner through a type l^1 diagonal. As we shall see this is not the case. Indeed, several new spaces of operators arise as natural extensions of the nuclear factorization result of Grothendieck. Moreover, these operators can be completely characterized by certain tensor product representations.

We first give a characterization of the diagonal operators from l^{∞} to l^1

which are of type l^1. Without loss of generality we may assume that for diagonals $T \sim (\lambda_i)_{i=1}^\infty$ we have $|\lambda_i| \geqslant |\lambda_{i+1}|$ for all i (see, e.g., [2], [6]).

PROPOSITION 1.1. *If* $T: l^\infty \to l^1$ *is a diagonal*, $T \sim (\lambda_i)_{i=1}^\infty$, *then* $\alpha_k(T) = \sum_{i=k+1}^\infty |\lambda_i|$ *for each* k.

A proof of (1.1) can be found in [2]. The next result now follows immediately.

THEOREM 1.2. *A diagonal* $T: l^\infty \to l^1$ *with* $T \sim (\lambda_i)_{i=1}^\infty$ *is of type* l^1 *if and only if* $\sum_{i=1}^\infty i|\lambda_i|$ *converges.*

It follows that if $T \in L(E, F)$ has a representation $T = \sum_{i=1}^\infty \lambda_i f_i \otimes y_i$ where $(f_i)_{i=1}^\infty \subset U_{E'}$, $(y_i)_{i=1}^\infty \subset U_F$ and $\sum_{i=1}^\infty i|\lambda_i|$ is finite then T factors through a diagonal from l^∞ to l^1 which is of type l^1. Indeed, we have

$$
\begin{array}{ccc}
E & \xrightarrow{\ T\ } & F \\
A \downarrow & & \uparrow B \\
l^\infty & \xrightarrow{\ \mathcal{D}\ } & l^1
\end{array}
\qquad .
$$

where $Ax = (\langle x, f_i \rangle)_{i=1}^\infty$, $\mathcal{D} \sim (\lambda_i)_{i=1}^\infty$ and $B(\xi_i)_{i=1}^\infty = \sum_{i=1}^\infty \xi_i y_i$. Since $\alpha_k(\mathcal{D}) \leqslant \sum_{i=k+1}^\infty |\lambda_i|$ we have $\sum_{k=1}^\infty \alpha_{k-1}(\mathcal{D}) \leqslant \sum_{k=1}^\infty \sum_{i=k}^\infty |\lambda_i| \leqslant \sum_{i=1}^\infty i|\lambda_i|$; hence $\mathcal{D}$ is of type l^1.

We next show that this representation actually characterizes those operators which factor through a diagonal of type l^1 from l^∞ to l^1.

DEFINITION 1.3. We will say that $T \in L(E, F)$ is l^1-*factorable* if T factors through a diagonal of type l^1 from l^∞ to l^1.

PROPOSITION 1.4. *An operator* $T \in L(E, F)$ *is* l^1-*factorable if and only if* T *has a representation* $T = \sum_{i=1}^\infty \lambda_i f_i \otimes y_i$ *where* $(f_i)_{i=1}^\infty \subset U_{E'}$, $(y_i)_{i=1}^\infty \subset U_F$ *and* $\sum_{i=1}^\infty i|\lambda_i|$ *converges.*

PROOF. We have already observed the necessity, so suppose that T has such a factorization. Then $T = B\mathcal{D}A$ where

$$
\begin{array}{ccc}
E & \xrightarrow{\ T\ } & F \\
A \downarrow & & \uparrow B \\
l^\infty & \xrightarrow{\ \mathcal{D}\ } & l^1
\end{array}
$$

with $\mathcal{D} \sim (\lambda_i)_{i=1}^\infty$. We may assume that $|\lambda_i| \geqslant |\lambda_{i+1}|$ for all i by a permutation of the indices if necessary. Since $\mathcal{D} \in l^1(l^\infty, l^1)$ it follows from (1.2) that $\sum_{i=1}^\infty i|\lambda_i|$ converges. Define $f_i \in E'$ by $f_i x = (1/\|A\|)\langle e_i, Ax \rangle$ where e_i is the ith unit vector and let $y_i = (1/\|B\|)Be_i$. If $\mu_i = \lambda_i \|A\| \|B\|$ then for $x \in E$ we

have

$$\sum_{i=1}^{\infty} \mu_i \langle x, f_i \rangle y_i = \sum_{i=1}^{\infty} \lambda_i \langle e_i, Ax \rangle Be_i = B \left(\sum_{i=1}^{\infty} \lambda_i \langle e_i, Ax \rangle e_i \right)$$

$$= B\mathcal{D}(\langle e_i, Ax \rangle)_{i=1}^{\infty} = B\mathcal{D}Ax = Tx;$$

hence $T = \Sigma_{i=1}^{\infty} \mu_i f_i \otimes y_i$ is the desired representation.

Clearly l^1-factorable operators are of type l^1. However, the two classes of operators are, in general, distinct. To see this we need the following theorem of H. Weyl [10] (see also [1, §2]).

THEOREM 1.5. *Let* H *be a Hilbert space and* $0 < p \leqslant 1$. *If* $T \in L^{(p)}(H, H)$ *and* $(z_i)_{i=1}^{\infty}$ *is the sequence of eigenvalues of* T, *arranged in order of decreasing modulus and repeated according to multiplicity, then* $\Sigma_{i=1}^{\infty} |z_i|^p \leqslant S_p(T)$.

Now let $T: l^2 \longrightarrow l^2$ be the diagonal $T \sim (i^{-3/2})_{i=1}^{\infty}$. That T is of type l^1 follows from [6], [7]. If T were l^1-factorable then by (1.4) there exist $(f_i)_{i=1}^{\infty} \subset U_{l^2}$, $(y_i)_{i=1}^{\infty} \subset U_{l^2}$ and scalars $(\mu_i)_{i=1}^{\infty}$ with $\Sigma_{i=1}^{\infty} i|\mu_i|$ finite such that $T = \Sigma_{i=1}^{\infty} \mu_i f_i \otimes y_i$. But then

$$\sum_{i=1}^{\infty} |\mu_i|^{2/3} = \sum_{i=1}^{\infty} (i |\mu_i|)^{2/3} i^{-2/3} \leqslant \left(\sum_{i=1}^{\infty} i |\mu_i| \right)^{2/3} \left(\sum_{i=1}^{\infty} i^{-2} \right)^{1/3} < +\infty;$$

hence $T \in L^{(2/3)}(l^2, l^2)$. That is, $S_{2/3}(T)$ is finite. But since the sequence of eigenvalues of T is precisely $(i^{-3/2})_{i=1}^{\infty}$ it follows from (1.5) that $S_{2/3}(T) \geqslant \Sigma_{i=1}^{\infty}(i^{-3/2})^{2/3} = \Sigma_{i=1}^{\infty} i^{-1}$. This contradiction implies that T is not l^1-factorable.

DEFINITION 1.6. We will say that $T \in L(E, F)$ is *p-factorable*, $p \geqslant 1$, if T factors through a diagonal $\mathcal{D}: l^{\infty} \longrightarrow l^1$ having the property that $\Sigma_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{D})$ converges. We will denote the collection of p-factorable operators from E to F by $F_p(E, F)$.

It is clear that $F_1(E, F)$ coincides with the l^1-factorable operators from E to F. In fact, (1.4) can be generalized in a natural way to the p-factorable operators.

THEOREM 1.7. *An operator* $T \in L(E, F)$ *is p-factorable for* $p \geqslant 1$ *if and only if there are sequences* $(f_n)_{n=1}^{\infty} \subset U_{E'}$, $(y_n)_{n=1}^{\infty} \subset U_F$ *and scalars* $(\lambda_n)_{n=1}^{\infty}$ *with* $\Sigma_{n=1}^{\infty} n^p |\lambda_n|$ *finite such that* $T = \Sigma_{n=1}^{\infty} \lambda_n f_n \otimes y_n$.

PROOF. If T has such a representation then $T = B\mathcal{D}A$ where B, $\mathcal{D}$ and A are as in (1.4); since $\alpha_n(\mathcal{D}) \leqslant \Sigma_{k=n+1}^{\infty} |\lambda_k|$ for each n we have

$$\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{D}) \leqslant \sum_{n=1}^{\infty} n^{p-1} \sum_{k=n}^{\infty} |\lambda_k| \leqslant \sum_{n=1}^{\infty} n^{p} |\lambda_n|;$$

hence $\mathcal{D}$ has the desired property.

Conversely, if T factors through a diagonal $\mathcal{D}$ having the property that $\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{D})$ converges then $\mathcal{D} \sim (\lambda_n)_{n=1}^{\infty}$ where $|\lambda_{n+1}| \geqslant |\lambda_n|$ for all n (we can assume $(\lambda_n)_{n=1}^{\infty}$ is decreasing by taking a suitable permutation of the indices if necessary). Then
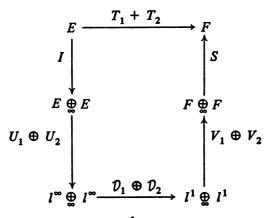
$$\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{D}) = \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=n}^{\infty} |\lambda_k| \right)$$

$$= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} i^{p-1} \right) |\lambda_n| \geqslant K(p) \sum_{n=1}^{\infty} n^{p} |\lambda_n|$$

where $K(p)$ is a constant depending only on p. Let f_i, y_i and μ_i be as in (1.4). Then $T = \sum_{i=1}^{\infty} \mu_i f_i \otimes y_i$ is a representation of T having the desired property.

DEFINITION 1.8. For $p \geqslant 1$ and $T \in F_p(E, F)$ let $f_p(T) = \inf\{\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{D}): \mathcal{D}$ is a diagonal from l^{∞} to l^1 with T factoring through $\mathcal{D}\}$.

THEOREM 1.9. *For every* $p \geqslant 1$, $F_p(E, F)$ *is a vector space.*

PROOF. Let $p \geqslant 1$ and $T_1, T_2 \in F_p(E, F)$. Then $T_i = V_i \mathcal{D}_i U_i$ where $\mathcal{D}_i: l^{\infty} \longrightarrow l^1$ is a diagonal with $\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{D}_i)$ finite for $i = 1, 2$. Define $I: E \longrightarrow E \oplus_{\infty} E$ by $Ix = (x, x)$ and $S: F \oplus_{\infty} F \longrightarrow F$ by $S(x, y) = x + y$. I and S are bounded linear transformations and we have the commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\quad T_1 + T_2 \quad} & F \\
\downarrow{\scriptstyle I} & & \uparrow{\scriptstyle S} \\
E \oplus_{\infty} E & & F \oplus_{\infty} F \\
\downarrow{\scriptstyle U_1 \oplus U_2} & & \uparrow{\scriptstyle V_1 \oplus V_2} \\
l^{\infty} \oplus_{\infty} l^{\infty} & \xrightarrow{\quad \mathcal{D}_1 \oplus \mathcal{D}_2 \quad} & l^1 \oplus_{\infty} l^1
\end{array}
$$

We will first show that $\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{D}_1 \oplus \mathcal{D}_2)$ converges. Indeed, if $A_i: l^{\infty} \longrightarrow l^1$ is of rank at most k with $\|\mathcal{D}_i - A_i\| < \alpha_k(\mathcal{D}_i) + \epsilon$, $i = 1, 2$, then the operator $A = A_1 \oplus A_2: l^{\infty} \oplus_{\infty} l^{\infty} \longrightarrow l^1 \oplus_{\infty} l^1$ has rank at most $2k$. If

$\xi = (\xi^1, \xi^2) \in l^\infty \oplus_\infty l^\infty$ and $\|\xi\|_\infty \leqslant 1$ then

$$\|(\mathcal{D}_1 \oplus \mathcal{D}_2 - A)\xi\|_\infty = \|(\mathcal{D}_1\xi^1 - A_1\xi^1, \mathcal{D}_2\xi^2 - A_2\xi^2)\|_\infty$$

$$= \max\{\|\mathcal{D}_1\xi^1 - A_1\xi^1\|_1, \|\mathcal{D}_2\xi^2 - A_2\xi^2\|_1\} < \max_{i=1,2} \alpha_k(\mathcal{D}_i) + \epsilon.$$

Thus $\alpha_{2k}(\mathcal{D}_1 \oplus \mathcal{D}_2) \leqslant \max_{i=1,2}\alpha_k(\mathcal{D}_i)$ for each k. Since $\alpha_{2k+1}(\mathcal{D}_1 \oplus \mathcal{D}_2) \leqslant \alpha_{2k}(\mathcal{D}_1 \oplus \mathcal{D}_2)$ and $(2k+1)^{p-1} \leqslant C(p)(2k)^{p-1}$ for all k it follows that $\Sigma_{k=1}^\infty k^{p-1}\alpha_{k-1}(\mathcal{D}_1 \oplus \mathcal{D}_2)$ converges if and only if $\Sigma_{k=0}^\infty (2k)^{p-1}\alpha_{2k}(\mathcal{D}_1 \oplus \mathcal{D}_2)$ converges. But

$$\sum_{k=0}^\infty (2k)^{p-1}\alpha_{2k}(\mathcal{D}_1 \oplus \mathcal{D}_2) \leqslant 2^{p-1}\max_{i=1,2}\left\{\sum_{k=0}^\infty (k+1)^{p-1}\alpha_k(\mathcal{D}_i)\right\}.$$

We next show that $\mathcal{D}_1 \oplus \mathcal{D}_2$ factors through a diagonal $\mathcal{D}: l^\infty \longrightarrow l^1$ having the property that $\Sigma_{n=1}^\infty n^{p-1}\alpha_{n-1}(\mathcal{D})$ converges. If $(\xi, \eta) \in l^\infty \oplus_\infty l^\infty$ define $R: l^\infty \oplus_\infty l^\infty \longrightarrow l^\infty$ by $R(\xi, \eta) = (\gamma_i)_{i=1}^\infty$ where

$$\gamma_i = \xi_{(i+1)/2} \quad \text{if } i \text{ is odd,}$$

$$= \eta_{i/2} \quad \text{if } i \text{ is even,}$$

for $\xi = (\xi_i)_{i=1}^\infty$ and $\eta = (\eta_i)_{i=1}^\infty$. That is, $R(\xi, \eta) = (\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_n, \eta_n, \cdots) \in l^\infty$. Then R is well defined, linear, and $\|R\| = 1$. Define $\mathcal{D}: l^\infty \longrightarrow l^1$ by $\mathcal{D} \sim (\delta_i)_{i=1}^\infty$ where

$$\delta_i = \lambda_{(i+1)/2} \quad \text{if } i \text{ is odd,}$$

$$= \mu_{i/2} \quad \text{if } i \text{ is even,}$$

with $\mathcal{D}_1 \sim (\lambda_i)_{i=1}^\infty$ and $\mathcal{D}_2 \sim (\mu_i)_{i=1}^\infty$. Now $\|\mathcal{D}\| \leqslant \Sigma_{i=1}^\infty (|\lambda_i| + |\mu_i|)$ so $\mathcal{D} \in L(l^\infty, l^1)$. Let $F: l^1 \longrightarrow l^1 \oplus_\infty l^1$ be defined by $F((\xi_i)_{i=1}^\infty) = ((\alpha_i)_{i=1}^\infty, (\beta_i)_{i=1}^\infty)$, $\alpha_i = \xi_{2i-1}$ and $\beta_i = \xi_{2i}$. It follows from the definitions that $\mathcal{D}_1 \oplus \mathcal{D}_2 = F\mathcal{D}R$; hence $T_1 + T_2$ factors through the diagonal $\mathcal{D}: l^\infty \longrightarrow l^1$. To see that $\Sigma_{n=1}^\infty n^{p-1}\alpha_{n-1}(\mathcal{D})$ is finite we need only observe that $\mathcal{D} = B(\mathcal{D}_1 \oplus \mathcal{D}_2)A$ where $A: l^\infty \longrightarrow l^\infty \oplus_\infty l^\infty$ is defined by $A((\xi_i)_{i=1}^\infty) = (\xi^1, \xi^2)$ with $\xi^1 = (\xi_1, \xi_3, \xi_5, \cdots, \xi_{2n+1}, \cdots)$ and $\xi^2 = (\xi_2, \xi_4, \xi_6, \cdots, \xi_{2n}, \cdots)$ and $B: l^1 \oplus_\infty l^1 \longrightarrow l^1$ is defined by $B((\xi_i)_{i=1}^\infty, (\eta_i)_{i=1}^\infty) = (\xi_1, \eta_1, \xi_2, \eta_2, \cdots)$. Now $\|A\| = 1$ and $\|B\| \leqslant 2$ so $\alpha_k(\mathcal{D}) \leqslant \|A\|\alpha_k(\mathcal{D}_1 \oplus \mathcal{D}_2)\|B\| \leqslant 2\alpha_k(\mathcal{D}_1 \oplus \mathcal{D}_2)$.

COROLLARY 1.10. *For* $p \geqslant 1$ *and* $T_i \in F_p(E, F)$, $i = 1, 2$, *there is a constant* $C(p)$ *depending only on* p *such that* $f_p(T_1 + T_2) \leqslant C(p)(f_p(T_1) + f_p(T_2))$. *If* $p = 1$ *we can take* $C(1) = 2$.

We now show that $F_p(E, F)$ is complete in the topology generated by f_p. The following lemma is the first step.

LEMMA 1.11. *If $p \geqslant 1$ and $D_p(l^\infty, l^1)$ denotes the space of all diagonals $D: l^\infty \to l^1$ such that $\sigma_p(D) = \sum_{n=1}^\infty n^{p-1} \alpha_{n-1}(D)$ is finite then $D_p(E, F)$ is complete under the topology generated by σ_p.*

PROOF. Suppose $\{D_n\}_{n=1}^\infty \subset D_p(l^\infty, l^1)$ is σ_p-cauchy. Then there is a diagonal $D: l^\infty \to l^1$ such that $\|D_n - D\| \to 0$. Thus $\alpha_k(D_n - D) \underset{n}{\to} 0$ uniformly in k. Let $M > 0$ be such that $\sigma_p(D_n) \leqslant M$ for all n. Then

$$\sum_{n=1}^N n^{p-1} \alpha_{n-1}(D) = \sum_{n=1}^N n^{p-1}(\alpha_{n-1}(D) + \alpha_{n-1}(D_m)) + \sum_{n=1}^N n^{p-1} \alpha_{n-1}(D_m).$$

If m is chosen so that $k^{p-1}|\alpha_{k-1}(D) - \alpha_{k-1}(D_m)| < 1/N$ for all k, $1 \leqslant k \leqslant N$, then $\sum_{n=1}^N n^{p-1}\alpha_{n-1}(D) < M + 1$. That is, $\sigma_p(D)$ is finite. That $\sigma_p(D - D_m) \underset{m}{\longrightarrow} 0$ is clear.

THEOREM 1.12. *For each $p \geqslant 1$, $F_p(E, F)$ is complete under the topology generated by f_p.*

PROOF. If $(T_n)_{n=1}^\infty \subset F_p(E, F)$ is f_p-cauchy then there is $T \in L(E, F)$ such that $\|T_n - T\| \to 0$. Choose a subsequence $(T_{n_i})_{i=1}^\infty$ of $(T_n)_{n=1}^\infty$ such that $f_p(T_{n_{i+1}} - T_{n_i}) < 2^{-8i}$ for each i and let $S_i = T_{n_{i+1}} - T_{n_i}$. Then $T = T_{n_1} + \sum_{i=1}^\infty S_i = \sum_{i=0}^\infty S_i$ where $S_0 = T_{n_1}$. Write $S_i = V_i D_i U_i$ with $\|V_i\| = \|U_i\| = 1$ and $\sum_{n=1}^\infty n^{p-1}\alpha_{n-1}(D_i) \leqslant f_p(S_i) + 2^{-8i}$. Then we can assume $\|V_i\| = \|U_i\| = 2^{-i}$ for each $i \geqslant 1$ and $\sum_{n=1}^\infty n^{p-1}\alpha_{n-1}(D_i) < 2^{-5i}$ and $\|V_0\| = \|U_0\| = 1$ with $\sum_{n=1}^\infty n^{p-1}\alpha_{n-1}(D_0) < f_p(S_0) + 2^{-6i}$. Then $\sum_{n=0}^\infty \|V_n\|$ and $\sum_{n=0}^\infty \|U_n\|$ converge so there are $U \in L(E, l^\infty)$ and $V \in L(l^1, F)$ such that $U = \sum_{n=0}^\infty U_n$ and $V = \sum_{n=0}^\infty V_n$. Also $\sum_{n=0}^\infty \sigma_p(D_n)$ converges and so by (1.11) there is $D \in D_p(l^\infty, l^1)$ with $D = \sum_{n=0}^\infty D_n$. Now

$$\left\| VDU - \sum_{i=0}^n V_i D_i U_i \right\| \leqslant \left\| \sum_{i=n+1}^\infty V_i P_i \right\| \left\| \sum_{i=n+1}^\infty D_i U_i \right\|$$

$$\leqslant \left\| \sum_{i=n+1}^\infty V_i P_i \right\| \left\| \sum_{i=n+1}^\infty D_i Q_i \right\| \left\| \sum_{i=n+1}^\infty U_i \right\|$$

$$\leqslant \sum_{i=n+1}^\infty \|V_i\| \sum_{i=n+1}^\infty \|D_i\| \sum_{i=n+1}^\infty \|U_i\| \underset{n}{\to} 0$$

where $P_i: (\sum_{i=1}^\infty \oplus l_i^1) \to l_i^1$ and $Q_i: (\sum_{i=1}^\infty \oplus l_i^\infty) \to l_i^\infty$ are the natural projections (viewing l^1 and l^∞ as $(\sum_{i=1}^\infty \oplus l_i^1)$ and $(\sum_{i=1}^\infty \oplus l_i^\infty)$ respectively). Thus $VDU = \sum_{i=0}^\infty S_i = T$.

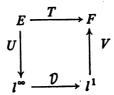**2. Relationships between various classes of operators.** In this section we

study the relationship between the spaces of operators $F_p(E, F), F_p(E, F),$ $L^{(p)}(E, F)$ and $l^p(E, F)$. We obtain several factorization theorems pertaining to operators of these various types.

It is clear from the definitions that $F_p(E, F) \subset F_{1/p}(E, F)$ for each $p \geqslant 1$ and $f_p(T) \geqslant F_{1/p}(T)$. It also follows immediately that for $0 < p \leqslant \frac{1}{2}, F_p(E, F)$ $\subset F_q(E, F)$ for every $q, 1 \leqslant q < (1 - p)/p$, and $f_q(T) \leqslant K(p)F_p(T)$ where $K(p)$ is a constant depending only on p. In fact, this is the best result possible. To see this let $T: l^1 \longrightarrow l^1$ be the diagonal $T \sim (\lambda_n)_{n=1}^{\infty}$ where $\lambda_n = n^{-1/p}[\ln(n + 1)]^{-1}$ for $0 < p \leqslant 1$. Then $n^{1/p}\lambda_n \longrightarrow 0$; hence $T \in F_p(l^1, l^1)$. If $T \in F_m(l^1, l^1), m = (1 - p)/p$, then $T = A\mathcal{D}B$ where $\mathcal{D}: l^{\infty} \longrightarrow l^1$ is a diagonal with the property that $\Sigma_{n=1}^{\infty} n^{m-1}\alpha_{n-1}(\mathcal{D})$ converges. In particular, since $\alpha_n(T) \leqslant \|A\| \|B\|\alpha_n(\mathcal{D}), \Sigma_{n=1}^{\infty} n^{m-1}\alpha_{n-1}(T)$ converges. But $\alpha_n(T) = \lambda_{n+1}$ [6], [7] for each n. This contradiction shows that $T \notin F_m(l^1, l^1)$ where $m = (1 - p)/p$.

Pietsch [6], [7] has shown that every $T \in l^p(E, F), 0 < p \leqslant 1$, has a representation $T = \Sigma_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ where $(f_i)_{i=1}^{\infty} \subset U_{E'}, (y_i)_{i=1}^{\infty} \subset U_F$ and $(\lambda_i)_{i=1}^{\infty} \in l^p$; that is $l^p(E, F) \subset L^{(p)}(E, F)$ for each $p, 0 < p \leqslant 1$. We now show that, in general, the opposite inclusion does not hold.

PROPOSITION 2.1. *Let* $0 < q \leqslant 1$ *and* $T \in L^{(q)}(E, F)$. *Then* T *factors through a diagonal from* l^{∞} *to* l^1 *which is of type* l^p, $1/p = 1/q - 1$. *In particular,* T *is of type* l^p, $1/p = 1/q - 1$, *and this is the best result possible. If* $0 < q \leqslant 2/3$ *then* T_a *is of type* l^p *where* $1/p = 1/q - 3/2$.

PROOF. If $T \in L^{(q)}(E, F)$ then $T = \Sigma_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ where $\|f_i\|, \|y_i\| \leqslant 1$, $|\lambda_i| \geqslant |\lambda_{i+1}|$ for all i, and $(\lambda_i)_{i=1}^{\infty} \in l^q$. Thus T has the factorization

$$E \xrightarrow{\quad T \quad} F$$
$$U \downarrow \qquad \qquad \uparrow V$$
$$l^{\infty} \xrightarrow{\quad \mathcal{D} \quad} l^1$$

where $Ux = (\langle x, f_i \rangle)_{i=1}^{\infty}, \mathcal{D} \sim (\lambda_i)_{i=1}^{\infty}$, and $V = \Sigma_{i=1}^{\infty} e_i \otimes y_i$. Now $\mathcal{D} = \mathcal{D}_2 \mathcal{D}_1$ where

$$l^{\infty} \xrightarrow{\quad \mathcal{D} \quad} l^1$$
$$\mathcal{D}_1 \downarrow \quad \nearrow \mathcal{D}_2$$
$$l^1$$

with $\mathcal{D}_1 \sim (\lambda_i^q)_{i=1}^{\infty}$ and $\mathcal{D}_2 \sim (\lambda_i^{1-q})_{i=1}^{\infty}$. Since $\alpha_k(\mathcal{D}_2) = |\lambda_{k+1}|^{1-q}$ [6], $\mathcal{D}_2$

is of type $l^{q/(1-q)}$; hence $T = V\mathcal{D}_2\mathcal{D}_1 U$ is also of type $l^{q/(1-q)}$.

If $q \leqslant 2/3$, write $\mathcal{D} = \mathcal{D}_3\mathcal{D}_2\mathcal{D}_1$ where

$$\begin{array}{ccc} l^\infty & \xrightarrow{\;\mathcal{D}\;} & l^1 \\ \mathcal{D}_1\Big\downarrow & & \Big\uparrow\mathcal{D}_3 \\ l^1 & \xrightarrow{\;\mathcal{D}_2\;} & l^2 \end{array}$$

with $\mathcal{D}_3 \sim (\lambda_i^{q/2})_{i=1}^\infty$, $\mathcal{D}_2 \sim (\lambda_i^{1-3q/2})_{i=1}^\infty$ and $\mathcal{D}_1 \sim (\lambda_i^q)_{i=1}^\infty$. Since $\alpha_k(\mathcal{D}_2) \leqslant |\lambda_{k+1}|^{1-3q/2}$ it follows that $\mathcal{D}_2$ is of type l^p, $1/p = 1/q - 3/2$. To finish the proof observe that $T_a = V\mathcal{D}_3P\mathcal{D}_2\mathcal{D}_1 U$ where $P \colon l^2 \to \overline{(V\mathcal{D}_3)^{-1}(T(E))}$ is the orthogonal projection.

We now show that the result $T \in L^{(q)}(E, F)$, $0 < q < 1$, implies that $T \in l^p(E, F)$, $1/p = 1/q - 1$, is the best result possible. Indeed, for $0 < q < 1$ we will construct a diagonal $T \colon l^\infty \to l^1$ which is strongly q-summable but not of type $l^{[q/(1-q)]-\epsilon}$ for any $\epsilon > 0$. For fixed q, $0 < q < 1$, choose β such that $\beta q > 1$. Let $\beta_n = (n + 1)^{(q-1)/q}[\ln(n + 1)]^{-\beta}$ and define $T \colon l^\infty \to l^1$, $T \sim (\lambda_n)_{n=1}^\infty$, where $\lambda_n = \beta_n - \beta_{n+1}$. It follows from the mean value theorem and the choice of β that $\sum_{n=1}^\infty |\beta_n - \beta_{n+1}|^q$ is finite; hence $T \in L^{(q)}(l^\infty, l^1)$. Since $\alpha_n(T) = \sum_{i=n+1}^\infty \lambda_i = \beta_{n+1}$ [2], T is not of type $l^{[q/(1-q)]-\epsilon}$ by the choice of β_n and β.

We point out that in [4] Markus proved that $L^{(p)}(E, F) \subset l^{p/(1-p)}(E, F)$ (using completely different techniques) and remarked without proof that $p/(1 - p)$ was the best result possible.

The following proposition is well known and easy to prove.

PROPOSITION 2.2. *If* $(\lambda_n)_{n=1}^\infty$ *is any sequence of positive scalars such that* $\sum_{n=1}^\infty n^p\lambda_n$ *converges for* $p \geqslant 1$ *then* $(\lambda_n)_{n=1}^\infty \in l^{(1+\epsilon)/(p+1)}$ *for every* $\epsilon > 0$ *and this is the best result possible. If* $(\lambda_n)_{n=1}^\infty \in l^{1/(p+1)}$ *for* $p \geqslant 1$ *then* $\sum_{n=1}^\infty n^p\lambda_n$ *converges and this is the best result possible.*

PROPOSITION 2.3. *If* $p > 1$ *and* $T \in F_p(E, F)$ *then* T *factors through a diagonal* $\mathcal{D} \colon l^1 \to l^2$ *of type* $l^{1/(p-1)}$. *In particular* T_a *is of type* $l^{1/(p-1)}$. *Moreover* T *factors through a diagonal* $\mathcal{D}_0 \colon l^1 \to l^1$ *of type* $l^{2/(2p-1)}$; *hence* $T \in l^{2/(2p-1)}(E, F)$.

PROOF. If $T \in F_p(E, F)$ then, by (1.7), $T = \sum_{n=1}^\infty \lambda_n f_n \otimes y_n$ where $\|f_n\|$, $\|y_n\| \leqslant 1$ and $\sum_{n=1}^\infty n^p|\lambda_n|$ converges. In particular, $(\lambda_n)_{n=1}^\infty \in l^{2/(2p+1)}$ by (2.2). Therefore we can factor T

$$\begin{array}{ccc} E & \longrightarrow & F \\ A\Big\downarrow & & \Big\uparrow B \\ l^\infty & \xrightarrow{\;\tilde{\mathcal{D}}\;} & l^1 \end{array}$$

where A, B and $\tilde{D}$ are as in (1.4). Now we can write $\tilde{D} = D_2 D D_3^{-}$ where

$$
\begin{array}{ccc}
l^\infty & \xrightarrow{\;\tilde{D}\;} & l^1 \\
D_3 \downarrow & & \uparrow D_2 \\
l^1 & \xrightarrow{\;D\;} & l^2
\end{array}
$$

with $D_3 \sim (\lambda_n^{2/(2p+1)})_{n=1}^\infty$, $D \sim (\lambda_n^{2(p-1)/(2p+1)})_{n=1}^\infty$ and $D_2 \sim (\lambda_n^{1/(2p+1)})_{n=1}^\infty$. Thus $\alpha_k(D) \leqslant |\lambda_{k+1}|^{2(p-1)/(2p+1)}$ and so $D \in l^{1/(p-1)}(l^1, l^2)$; it follows that $T_a \in l^{1/(p-1)}(E, \overline{T(E)})$. If we let $D_0 = D_2 D$ then $D_0 \sim (\lambda_n^{(2p-1)/(2p+1)})_{n=1}^\infty$. Since $\alpha_k(D_0) \leqslant |\lambda_{k+1}|^{(2p-1)/(2p+1)}$ we have the desired result.

If $p = 1$ we have the following version of (2.3).

PROPOSITION 2.4. *If $T \in F_1(E, F)$ then T_a is of type l^2. Indeed, T factors through a diagonal from l^∞ to l^2 which is of type l^2.*

PROOF. For $T \in F_1(E, F)$ choose a representation $T = \Sigma_{n=1}^\infty \lambda_n f_n \otimes y_n$ with $\|f_n\|$, $\|y_n\| \leqslant 1$ and $\Sigma_{n=1}^\infty n|\lambda_n|$ finite. Then $T = VDU$ where

$$
\begin{array}{ccc}
E & \xrightarrow{\;T\;} & F \\
U \downarrow & & \uparrow V \\
l^\infty & \xrightarrow{\;D\;} & l^2
\end{array}
$$

with $Ux = (\langle x, f_n \rangle)_{n=1}^\infty$, $D \sim (\lambda_n^{2/3})_{n=1}^\infty$, and $V = \Sigma_{n=1}^\infty \lambda_n^{1/3} e_n \otimes y_n$. The operators U, V and D are well defined by (2.2). Now $D = BA$ where

$$
\begin{array}{ccc}
l^\infty & \xrightarrow{\;D\;} & l^2 \\
{}_A \searrow & & \nearrow {}_B \\
& l^2 &
\end{array}
$$

and $A \sim (\lambda_n^{1/3})_{n=1}^\infty$ and $B \sim (\lambda_n^{1/3})_{n=1}^\infty$. Since B is of type l^2 [6], D is also of type l^2 and the result follows.

We point out that there are operators whose astrictions are of type l^2 yet the operators are not 1-factorable. Indeed, choose any Hilbert-Schmidt operator on Hilbert space which is not of type l^1. Then its astriction is also Hilbert-Schmidt, hence of type l^2 [6], but the operator is certainly not 1-factorable.

We next consider the relationship between p-factorable and strongly p-summable operators.

PROPOSITION 2.5. *Let $p \geqslant 1$ and $T \in L^{(q)}(E, F)$ where $q = 1/(p + 1)$. Then T is p-factorable. In particular, if T is of type l^q, $q = 1/(p + 1)$, then*

T is p-factorable and these are the best results possible.

The proof is immediate from (2.2).

It follows from (1.6) that for p-factorable operators T, $p \geq 1$, $\Sigma_{n=1}^{\infty} n^{p-1}\alpha_{n-1}(T)$ converges; hence, by (2.2), T is of type l^q for every $q > 1/p$. We next show that these are the best results possible.

PROPOSITION 2.6. *If $p \geq 1$ and T is p-factorable then $\Sigma_{n=1}^{\infty} n^{p-1}\alpha_{n-1}(T)$ converges and this is the best result possible. Moreover, T is of type $l^{(1/p)+\epsilon}$ for every $\epsilon > 0$ and, for $p > 1$, this is the best result possible.*

PROOF. The first part of the proposition follows from the definition of the p-factorable operators and the fact that for $T = B \mathcal{D} A$, $\alpha_k(T) \leq \|A\| \|B\| \alpha_k(\mathcal{D})$.

If $p > 1$ let $\beta_n = n^{-p}\ln^{-2}(n+1)$ and $\lambda_n = \beta_n - \beta_{n+1}$. Define T: $l^{\infty} \rightarrow l^1$ by $T \sim (\lambda_n)_{n=1}^{\infty}$. It follows from the mean value theorem that $\Sigma_{n=1}^{\infty} n^p |\lambda_n|$ converges. Since $\alpha_n(T) = \Sigma_{k=n+1}^{\infty} |\lambda_k|$ we have $\Sigma_{n=1}^{\infty} n^{p-1}\alpha_{n-1}(T)$ finite. But

$$\alpha_n(T) = \sum_{k=n+1}^{\infty} |\lambda_k| = \beta_{n+1} \quad \text{and} \quad \sum_{n=1}^{\infty} n^{(p-1)+\epsilon}\alpha_{n-1}(T) = \sum_{n=1}^{\infty} \frac{n^{(p-1)+\epsilon}}{(n+1)^p\ln^2(n+1)}$$

which diverges for any $\epsilon > 0$.

Now let $p > 1$ and $T \in F_p(E, F)$. It follows from the above and (2.2) that T is of type $l^{(1/p)+\epsilon}$ for every $\epsilon > 0$. To see that this is the best result possible let $\beta_n = n^{-p}\ln^{-p}(n+1)$, $\lambda_n = \beta_n - \beta_{n+1}$ and define $T: l^{\infty} \rightarrow l^1$ by $T \sim (\lambda_n)_{n=1}^{\infty}$. It again follows that $\Sigma_{n=1}^{\infty} n^p \lambda_n$ converges; hence $T \in F_p(l^1, l^{\infty})$. But $\alpha_k(T) = \Sigma_{n=k+1}^{\infty} \lambda_n = \beta_{k+1}$ and $(\beta_k)_{k=1}^{\infty} \notin l^{1/p}$; hence T is not of type $l^{1/p}$.

We point out that for $p = 1$ the result $F_p(E, F) \subset l^{(1+\epsilon)/p}(E, F)$ is not the best possible. Indeed, the first part of (2.6) gives $F_1(E, F) \subset l^1(E, F)$.

As a partial converse to (2.6) we have the following proposition.

PROPOSITION 2.7. *Let $p \geq 1$ and $T \in L(E, F)$. If $\Sigma_{n=1}^{\infty} n^p\alpha_{n-1}(T)$ converges then there are sequences $(f_n)_{n=1}^{\infty} \subset U_{E'}$, $(y_n)_{n=1}^{\infty} \subset U_F$ and scalars $(\lambda_n)_{n=1}^{\infty}$ with $\Sigma_{n=1}^{\infty} n^q |\lambda_n|$ finite for every $q, 0 < q < p$, such that $T = \Sigma_{n=1}^{\infty} \lambda_n f_n \otimes y_n$.*

PROOF. Since $\Sigma_{n=1}^{\infty} n^p\alpha_{n-1}(T)$ converges, $(\alpha_n(T))_{n=0}^{\infty} \in l^r$ for every $r > 1/(p+1)$ by (2.2); in particular $T \in l^r(E, F)$ for $1/(p+1) < r \leq 1$ and so by [6] there are sequences $(f_n)_{n=1}^{\infty} \subset U_{E'}$, $(y_n)_{n=1}^{\infty} \subset U_F$ and a nonincreasing sequence of scalars $(\lambda_n)_{n=1}^{\infty} \in l^r$ such that $T = \Sigma_{n=1}^{\infty} \lambda_n f_n \otimes y_n$. Now

$(n^{1/r}|\lambda_n|)^\infty_{n=1}$ is bounded; hence $\Sigma^\infty_{n=1} n^{1/r}|\lambda_n| n^{-(1+\epsilon)} = \Sigma^\infty_{n=1} n^{\dot{p}-\delta} |\lambda_n|$ converges for $\delta = \delta(\epsilon, r) > 0$.

If $T \in F_p(E, F)$, $S_1 \in L(G, E)$ and $S_2 \in L(F, X)$ then TS_1 and $S_2 T$ are p-factorable. For p-factorable T and q-factorable S, $p, q \geq 1$, we have the following composition formula.

PROPOSITION 2.8. *For* p, q, S *and* T *as above, the composition* ST *is r-factorable for every* r, $1 \leq r < p + q - 1$.

PROOF. We can write $T = \Sigma^\infty_{n=1} \lambda_n f_n \otimes y_n$ with $\|f_n\|$, $\|y_n\| \leq 1$ and $\Sigma^\infty_{n=1} n^p |\lambda_n|$ finite. The operator S also has a representation $S = \Sigma^\infty_{n=1} \mu_n g_n \otimes x_n$ with $\|g_n\|$, $\|x_n\| \leq 1$ and $\Sigma^\infty_{n=1} n^q |\mu_n|$ finite. Since $\alpha_n(S) \leq \Sigma^\infty_{i=n+1} |\mu_i|$ we have

$$n^q \alpha_{n-1}(S) \leq n^q \sum_{i=n}^\infty |\mu_i| \leq \sum_{i=n}^\infty i^q |\mu_i|;$$

thus $n^q \alpha_{n-1}(S) \to 0$. By (2.6), $\Sigma^\infty_{n=1} n^{p-1} \alpha_{n-1}(T)$ converges; hence

$$\sum_{n=1}^\infty n^{p+q-1} \alpha_{n-1}(ST) \leq \sum_{n=1}^\infty (2n)^{p+q-1} \alpha_{2n}(ST) + \sum_{n=1}^\infty (2n+1)^{p+q-1} \alpha_{2n+1}(ST)$$

$$\leq K(p, q) \sum_{n=1}^\infty n^{p+q-1} \alpha_{2n}(ST)$$

$$\leq K(p, q) \sum_{n=1}^\infty (n^{p-1} \alpha_{n-1}(T))(n^q \alpha_{n-1}(S)) < \infty.$$

By (2.7), ST has a representation $ST = \Sigma^\infty_{n=1} \beta_n h_n \otimes z_n$ where $\|h_n\|$, $\|z_n\| \leq 1$ and $\Sigma^\infty_{n=1} n^r |\beta_n|$ converges for every r, $1 \leq r < p + q - 1$. The result now follows from (1.7).

We now give a summary of this section.

THEOREM 2.9 (SUMMARY). (i) *If* $0 < p < 1$ *then*

$$L^{(p)}(E, F) \subset l^{p/(1-p)}(E, F)$$

and this is the best result possible.

(ii) *For* $0 < p \leq 1$, $l^p(E, F) \subset L^{(p)}(E, F)$.

(iii) *For* $p \geq 1$, $L^{(q)}(E, F) \subset F_p(E, F)$, $q = 1/(p + 1)$. *In particular,* $l^q(E, F) \subset F_p(E, F)$.

(iv) *For* $p \geq 1$, $F_p(E, F) \subset L^{(r)}(E, F)$ *for every* r, $1 \geq r > 1/(p + 1)$ *and this is the best result possible.*

(v) *For* $p > 1$, $F_p(E, F) \subset l^q(E, F)$ *for every* $q > 1/p$ *and this is the best result possible. If* $p = 1$ *we have* $F_1(E, F) \subset l^1(E, F)$ *and this is the best result possible.*

(vi) *For* $p \geqslant 1$, $F_p(E, F) \subset F_{1/p}(E, F)$.

(vii) *For* $0 < p \leqslant \frac{1}{2}$, $F_p(E, F) \subset F_q(E, F)$ *for every* q, $1 \leqslant q < (1 - p)/p$ *and this is the best result possible.*

(viii) *If* $0 < q \leqslant \frac{1}{2}$ *then* $L^{(q)}(E, F) \subset F_r(E, F)$ *for* $r = q/(1 - q)$. *In particular,* $l^q(E, F) \subset F_r(E, F)$.

(ix) *For* $0 < p \leqslant \frac{1}{2}$, $F_p(E, F) \subset L^{(r)}(E, F)$ *for every* r, $p/(1 - p) < r \leqslant 1$.

If we let $l^p_R(E, F)$ *denote the collection of operators from* E *to* F *whose astrictions are of type* l^p *then* (2.3) *and* (2.4) *give the following result.*

(x) *For* $p > 1$, $F_p(E, F) \subset l^{1/(p-1)}_R(E, F)$ *and for* $p = 1$, $F_1(E, F) \subset l^2_R(E, F)$.

We feel a few words about the class of operators $l^p_R(E, F)$ are in order. It is not difficult to give examples of operators T and spaces E, F for which $\alpha_k(T) \neq \alpha_k(T_a)$. For example, if $T: l^1 \longrightarrow l^\infty$ is natural injection then $\alpha_k(T) = \frac{1}{2}$ and $\alpha_k(T_a) = 1$ for all k [2]. It is not known if the astriction of an operator of type l^p is again of type l^p. We suspect this is not the case; however, we have been unable to construct such an example, even for $p = 1$. Results of Retherford and Stegall [9] on fully nuclear operators and of Grothendieck [1, §2] on the astrictions of strongly *p*-summable operators support our conjecture. We have shown [3] that the astriction of an operator of type $l^{2/3}$ is nuclear (hence such an operator is fully nuclear [9]) and that the astriction of an operator of type l^p, $0 < p \leqslant 1$, is of type l^r, $1/r = 1/p - 1$.

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